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Eduard Stiefel's linear programming method as tool in agro metrics

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Abstract. In this paper, we consider the linear optimization models' application problems in the research processes and in the didactics processes. Our target is to convince the colleagues about preferences of Eduard Stiefel's method comparing with widespread George Bernard Dantzig's simplex method. Indeed, the Stiefel's method provides researchers and teachers with clear and pithy interpretations of linear model. Our pedagogical praxis during long time period conclusively confirms that Stiefel's method makes the theory of linear optimization match easier for understanding and for active employing to students especially in the specialities with limited mathematical education. We offer in this paper also some new theoretical concepts and methods adapted for the linear model information analysis (the concept of general optimal plan, the methods of the profounder sensitivity analysis), and we appeal economists to interpret simplex predicates as productions functions in a broad sense.

Key words: linear programming, simplex method, Stiefel's method.

INTRODUCTION

Linearity postulate often holds in the relations between economic, financial, technological indicators. For example, financial accounting and managerial accounting mostly use linear functions. That is the reason why in spite of relative simplicity linear models allow us to describe interdependence between indicators in the different scientific areas rather adequately and satisfactorily for practical application. As result, the linear programming or linear planning (LP) is one of the most widely applied quantitative decision making approach techniques in Management Science. Since the very beginning linear programming was successfully applied also in agro metrics. Well-known and included in the education courses are such linear programming models of agriculture as land utilizing planning problem, problem of the rational structure of the live-stock breeding, problem of the rational nutrition for domestic animals.

Let us mention only two lately examples published in the international Journal 'Agronomy Research'. Significant model 'Optimization of arable land use to guarantee food security in Estonia' was elaborated by Põldaru, R, Viira, A.H. & Roots, J. (2018). The authors point out: 'The supply side of the model is a typical agricultural production model that guarantees the consistency of crop and livestock farming. The objective of the model is to minimize the use of arable land for field crops to ensure fodder for animal feed, and food for human food consumption. The model is used to analyse various land use strategies'.

The linear programming can be applied in different modified forms. For example, Žgajnar & Kavčič (2009) offer 'Multi-goal pig ration formulation' using weighted goal programming supported by a system of penalty functions.

In current paper, we more rigorously will analyse the model 'Decision Making in Agriculture with Linear Programming Approach' presented by N. A. Sofi, Aquil Ahmed, Mudasir Ahmad & Bilal Ahmad Bhat (2015). The authors assert that 'Linear programming technique is relevant in optimization of resource allocation and achieving efficiency in production planning particularly in achieving increased agriculture production of food crops (rice, maize, wheat, pulses, and other crops)'. The authors applied linear programming technique to determine the optimum land allocation of 5 food crops by using agriculture data, with respect to various factors for the period 2004–2011. This model seems to us appropriate example in order to demonstrate shortly the comparative advantages of the Stiefel's method and some innovations of authors as well.

The fact that the linear models satisfactorily adequately reflect real links between indicators is a crucial reason for wide applications of linear optimal planning. Besides that linear programming owes its popularity due to George Bernard Dantzig's simplex method (Dantzig, 1949; Dantzig, 1951; Dantzig et al., 1955; Dantzig, 1963; Cottle & Dantzig, 1968; Cottle & Dantzig, 1970; Dantzig & Thapa, 1977; Dantzig & Thapa, 2003) extensively embodied in the efficient software. For example, handy tool 'Analytic Solver Upgrade' (formerly 'Premium Solver Pro') solves larges problems – up to 2,000 variables 100 times faster than the standard Solver. By the way the problem 'Optimization of arable land use to guarantee food security in Estonia' (contains 163 variables and 178 constraints in form of linear equations) was solved by authors with help of 'Premium Solver Pro'.

Observation. We have suspicion that paper (Põldaru et al., 2018) contains a fallacy. If linear model contains 163 variables and 178 constraints in form of linear equations than there is a big chance to have empty set of feasible solutions because rank of system's matrix is less or equal 163. If feasible solution exists than equations of system are linear dependent and it is worth to investigate connections between constraints.

Therefore, linear programming is one of the most successful tools to implement quantitative approaches to management decision making. A large number of applications has been published in various industries including agro metrics. Let us mention such models as Production Scheduling, Multiperiod Production and Inventory Planning, Work Force Assignment and Staff Scheduling, Environmental problems, Transportation, Assignment and Transshipment problems, Blending Problems what occur, for example, in the food industry. In our opinion, the very significant role linear programming plays through its connection with Input-Output analysis. For example, Data Envelopment Analysis (DEA) is used to compare the relative efficiency of operating units whose input and output vectors have identical structure. We must mention the Goal programming and Multicriteria decision problems. High actuality keeps classical problem of Tchebycheff Approximation in case when linear model has not feasible solutions.

The classic of Mathematical Economics professor of London School of Economics R. G. D. Allen already in 1956 in the world famous book (Allen, 1956) 'Mathematical Economics' wrote about linear model (chapter 'Marginal Analysis v. Linear Programming of the Firm', 620 page): 'The linear programming approach seems very well adapted for application to decision-taking at the level of the firm. It provides, trough emphasis on technology, just the link required between the problems of interest to the economist and those which engage the attention of entrepreneur and engineers'.

So, there exist a lot of conceptual models in different areas of management science. No doubt that proper academic course can be formed as extremely rich and interesting for students because students with help of Microsoft Solver study simulated virtual problems. But for all that, each researcher perfectly knows that sufficient difficulties arise in the practical implementation of mentioned conceptual models. In all scientific conferences we took part the researchers agreed that the estimation of the linear expressions' coefficients as a rule is the most difficult task in the construction of the relevant mathematical model.

Generally speaking, the relevant linear programming models are expensive. Therefore self-evident is the desire to obtain from constructed model as far knowledges as possible.

How to obtain more knowledge about problem utilizing expensive linear programming model?

A wide overview of scientific and educational literature persuasively shows that G. B. Dantzig's simplex method till the nowadays is the mainstream method for solving linear programming problems. The mentioned linear programming applications in 'Agronomy Research' also are made with help of Dantzig's method. In the same time already in the year 1960 Professor of Mathematics Eduard Stiefel (Swiss Federal Institute of Technology in Zurich) in the article (Stiefel, 1960) 'Stiefel, E., Note on Jordan Elimination, Linear Programming and Tchebycheff Approximation, *Numerische Mathematik*, Vol. 2, 1960, 1–17)' offered another approach to the investigation of linear programming problem based on pivot transformations of the system of linear equations.

The goal of this paper is to conduct the comparative analysis of two different linear programming solving and analysis methods: well-known Dantzig's simplex method and Eduard Stiefel's method. We are going to explore the comparative advantages of Stiefel's method and demonstrate that Stiefel's method furnishes more information easily obtained from linear programming model. Moreover, on the ground of Stiefel's method we offer new concepts and constructive approaches in the linear problem investigation with help of linear programming. The new approaches are illustrated through five applications.

MATERIALS AND METHODS

Pivot transformation (often called as Jordan-Gauss elimination) is algorithmized equivalent transformation of the system of linear equations and simultaneous equivalent transformation of corresponding dual system of linear equations. Both are interpreted as predicates. Idea of pivot transformation as simultaneous equivalent transformation of two pairs of predicates is absolutely simple but incredibly fruitful in linear algebra.

Theorem. Pivot transformation (Jordan-Gauss elimination) as simultaneous equivalent transformations of two pairs of predicates: direct and dual.

Let **E**, **F** vector spaces. Let $a, b, c, d \in \mathbf{R}$; $a \neq 0$; $X_1, X_2, Y_1, Y_2 \in \mathbf{E}$; $U_1, U_2, V_1, V_2 \in \mathbf{F}$. Direct system of equations $\{ Y_1 = a X_1 + b X_2; Y_2 = c X_1 + d X_2 \}$ can be transformed as system $\{ X_1 = a^{-1} Y_1 - b a^{-1} X_2; Y_2 = c a^{-1} Y_1 + (a d - b c) a^{-1} X_2 \}$.

Dual system of equations { $U_1 = a\ V_1 + c\ V_2$; $U_2 = b\ V_1 + d\ V_2$ } can be transformed as system { $-V_1 = a^{-1}\ (-U_1) + c\ a^{-1}\ V_2$; $U_2 = b\ a^{-1}\ (-U_1) + (a\ d - b\ c)\ a^{-1}\ V_2$ }.

Proof of this theorem is omitted because it is based only to the elementary algebraic transformations of equations' systems.

We offer to consider these four systems of equations as predicates. Condition $a \neq 0$ is sufficient and necessary for equivalency of direct system and it's transformed system with respect to variables a, b, c, d $\in$ **R**; $X_1, X_2, Y_1, Y_2 \in$ **E**, and for equivalency of dual system and it's transformed system with respect to variables a, b, c, d $\in$ **R**; $U_1, U_2, V_1, V_2 \in$ **F** as well.

It is handy to write both systems in a table form (Table 1).

Table 1. Pivot transformation as algorithmized equivalent transformation

	X_1	X_2	
Y_1	a	b	V_1
Y_2	c	d	V_2
	U_1	U_2	

	Y_1	X_2	
X_1	a^{-1}	$-b\ a^{-1}$	$-U_1$
Y_2	$c\ a^{-1}$	$(a\ d - b\ c)\ a^{-1}$	V_2
	$-V_1$	U_2	

Remark. We demonstrated just (2×2) – matrix in order to be simple. Of course, number of vectors $X_i \in$ **E** and number of vectors $Y_j \in$ **F** are arbitrary.

Idea of Eduard Stiefel − goal-directed equivalent transformations (so called pivot transformations) of simplex predicate. What was reaction?

Let us observe, that articles of Dantzig & Thapa (Dantzig & Thapa, 1977; Dantzig & Thapa, 2003) contain references to the Hestenes & Stiefel's paper (Hestenes & Stiefel, 1952). The method of Eduard Stiefel was positively appraised by Albert Tucker and Michel Balinski, and widely used in Princeton University (Tucker, 1962; Balinski & Tucker, 1969). The article (Balinski & Tucker, 1969) 'Balinski, M.L., Tucker, A.W., Duality Theory of Linear Programs: A Constructive Approach with Applications, *SIAM Review*, Vol. 11, No. 3 (Jul., 1969), 347−377' contains the reference to the article of Eduard Stiefel (Stiefel, 1960).

In the Baltic States method of Eduard Stiefel mostly was known during the book (Zukovitskii & Avdeeva*,* 1967) 'Зуховицкий С. И., Авдеева Л. И. Линейное и выпуклое программирование. (Серия 'Экономико-математическая библиотека') Москва, 1967'. In the preface of this book Зуховицкий С. И. wrote: 'Вычислительным аппаратом в этой книге служит аппарат жордановых исключений, большие удобства которого убедительно продемонстрированы в статье Э. Штифеля 'Stiefel, E., Note on Jordan Elimination, Linear Programming and Tchebycheff Approximation, *Numerische Mathematik*, Vol. 2, 1960'.

Andrejs Jaunzems was active supporter of the Stiefel's method in Latvia (see, for example, the books (Jaunzems, 1981; Jaunzems, 1990; Jaunzems, 1993; Jaunzems, 2011; Jaunzems, 2013). In the book Jaunzems (1993) 'Mathematics for Economic Sciences. General course' theory of linear operator was based on the pivot transformations. Professor of Latvia University of Agriculture Alberts Krastiņš, which was good friend of Leonid Kantorovich (winner of the Stalin Prize in 1949 and the Nobel Memorial Prize in Economics in 1975), supported Stiefel's method through for a long time period widely used book (Krastiņš, 1976) 'Alberts Krastiņš. Matemātiskā programmēšana'.

In spite of the sufficient advantages of Stiefel's LP method comparing with Dantzig's method the method of Eduard Stiefel due to different reasons (probably, sometimes even due to peculiar reasons) does not belong to mainstream and is rarity in science and education.

For example, carefully elaborated teaching text (Border, 2004) ‘Border, K. M., The Gauss–Jordan and Simplex Algorithms’ published in California Institute of Technology utilizes traditional Dantzig’s method instead Stiefel’s LP method what is matched more suitable for teaching in the Division of the Humanities and Social Sciences. The characteristic example is also the book (Seo, 1991) ‘Seo, K.K. Managerial economics. Text, problems, and short cases, – Seventh edition, Irwin, 1991’ where solving of the extreme simple LP problem with help of Dantzig’s simplex method is demonstrated on the four (!) pages. In the same time, this book is presented as one of the most popular MBA textbooks of managerial economics in USA.

RESULTS AND DISCUSSION

Since a long period of time, we employ in our scientific and didactic praxis five kinds of Stiefel’s linear programming method applications what partly contain innovation elements. We would like to hope that our experience can be useful for many teachers and researchers.

Application 1. Duality.

Application 2. The concept of general optimal plan.

Application 3. Direct and dual simplex predicate as production functions.

Application 4. Sensitive analysis. Serious criticism of Microsoft Solver.

Application 5. Vectorial form.

Application 1. Duality

Let us examine LP standard model and it’s dual model.

max $\{C \cdot X \mid AX \le B, X \ge O\}$ or max $\{C \cdot X \mid U = -AX + B, X \ge O, U \ge O\}$;

$A \in \mathbf{R}^{m,\, n}$; $X, C \in \mathbf{R}^{n,\, 1}$; $U, B \in \mathbf{R}^{m,\, 1}$.

min $\{B \cdot Y \mid A^TY \ge C, Y \ge O\}$ or min $\{B \cdot Y \mid V = A^TY - C, Y \ge O, V \ge O\}$;

$Y \in \mathbf{R}^{m,\, 1}$, $V \in \mathbf{R}^{n,\, 1}$.

Let **X** and **Y** are the corresponding sets of feasible solutions for direct and dual problem. To solve those problems means to find the corresponding sets of optimal plans:

$\mathbf{X}^* := \{ X^* \mid X^* \in \mathbf{X}, C \cdot X^* \ge C \cdot X \ \forall X \in \mathbf{X} \}$.

$\mathbf{Y}^* := \{ Y^* \mid Y^* \in \mathbf{Y}, B \cdot Y^* \le B \cdot Y \ \forall Y \in \mathbf{Y} \}$.

Both systems of equation can be inscribed in the initial simplex table (Table 2).

Table 2. Direct and dual linear problems in the initial table form

	X	1	
U	−A	B	Y
z	C	0	1
	−V	W	

The direct problem is inscribed horizontally:

$U = -AX + B$, $z = C \cdot X + 0 \cdot 1$.

The dual problem is inscribed vertically:

$-V = -A^TY + C$, $w = B \cdot Y + 0 \cdot 1$.

We must take in account that

$X \ge O$, $U \ge O$; $Y \ge O$, $V \ge O$.

In order to investigate in a versatile manner this optimization problem we must make goal-oriented pivot transformations what mean simultaneous equivalent transformations of both systems – direct and dual. Duality means, that each pivot transformation of the system: $u_i \uparrow\downarrow x_j$, $x_k \uparrow\downarrow u_l$, $u_i \uparrow\downarrow u_j$, $x_k \uparrow\downarrow x_l$ simultaneously is the **modified** pivot transformation of the dual system: $v_j \uparrow^m\downarrow y_i$, $y_l \uparrow^m\downarrow v_k$, $y_j \uparrow^m\downarrow y_i$, $v_l \uparrow^m\downarrow v_k$.

Our target is to obtain such table, where z expression contains only non-positive coefficients but w expression contains only non-negative coefficients. Regular case is when coefficients in both expressions are not equal zero. Then both problems have unique solution because z has maximal value and w has minimal value then and only then if all variables in the final expressions of z and w equals zero.

Application 2. The concept of general optimal plan

If z expression in the final table contains some zero coefficients than z can take maximal value also by non-zero values of proper variables. In this case set of optimal solutions of the direct problem **X*** (generally speaking) is infinite. If w expression contains some zero coefficient than set of optimal solutions of the dual problem **Y*** (generally speaking) is infinite. If z expression and w expression both contains some zero coefficients than sets **X***, **Y*** are infinite.

We never meet the concept of general optimal plan in the literature available for us. Therefore, until the opposite is not proved, the concept of general optimal plan first is offered in the paper (Jaunzems, 2013): 'Jaunzems A. Singulārā lineārā programmēšana menedžmenta ekonomikā'. Let us examine application in the managerial accounting: the break even set in case of multiproduct output. The modified simple example from book (Coenenberg, 1997) 'Adolf G. Coenenberg. Kosten-rechnung und Kostenanalyse' is used.

The start of modified citation:

Example. Two products P_1, P_2, two machines with working time limits 200, 300 hours. Contribution margins are 300 DM, 280 DM. Fixed costs = 8,400 DM. Find the break even set.

Product	P_1	P_2
First machine (hours per unit of product)	2	1
Second machine (hours per unit of product)	2	3

The model is: $300\,x_1 + 280\,x_2 \to max$ with respect the constraints $2\,x_1 + 1\,x_2 \le 200$, $2\,x_1 + 3\,x_2 \le 300$, $x_1 \ge 0$, $x_2 \ge 0$. The end of modified citation.

We offer the simplex predicate method. Table 3 shows the initial system, Table 4 shows the final system of equations.

Table 3. Initial table

	x_1	x_2	1
u_1	-2	-1	200
u_2	-2	-3	300
CM−FC	300	280	-8,400
CM	300	280	0

CM − contribution margin, FC − fixed costs.

Table 4. Final table. Break even set

	CM−FC	x_1	1
u_2	-0.0107	1.2143	210
x_2	0.0036	-1.0714	30
u_1	-0.0036	-0.9286	170
CM	1	0	8,400

We get points on the break even set, if $x_1 \ge 0$, $x_2 \ge 0$; $u_1 \ge 0$, $u_2 \ge 0$, CM − FC = 0.
The break even set predicate is the system of equations:
$u_1 = -0.929\,x_1 + 170$; $u_2 = 1.214\,x_1 + 210$, $x_2 = -1.071\,x_1 + 30$, $0 \le x_1 \le 28$.
Answers to different 'what if' questions.
For example, let us assume, that CM − FC = 0, $x_1 = 10$.

Then $x_2 = 19.29$; $u_1 = 160.71$; $u_2 = 222.14$.
The point (10; 19.29) belongs to the break even set.

Application 3. Direct and dual simplex predicate as production functions

In order to be clear and short we utilize agro metrics example from article (Sofi et al., 2015). We are not responsible for quality of this article more or less correctly reflecting optimal land structure planning problem. We are going with help of this example to show Stiefel's method in action. The authors of this article tried to find the optimal land utilizing structure in order to get the maximum output of the major food crops under land, capital and two kinds of labour availability constraint. In this model the 5 variables (rice, maize, wheat, pulses and other crops) are included. The authors used linear programming – simplex method. We decided to utilize the following long citation from article (Sofi et al., 2015) as example to explain the essence of applications 3, 4, and 5.

The start of citation: The objective function is the output of various agriculture productions of food crops, inequalities is the Land / Capital/Labour (A) and Labour (B) and requirement is total. Now, our objective is to find the optimum land of food crops. Table 5 represents in simplified manner the basic information necessary in order to construct a linear programming model of land utilization.

Table 5. Output per acre and the requirements

Variable food crops	Output/acre Lakh	Land (acre)	Capital/acre	Labour (A) Man day	Labour (B) working hours
Rice	102.85	773.40	22.03	10.08	13.09
Maize	114.84	941.84	45.74	14.10	12.07
Wheat	263.50	823.13	63.47	17.33	18.08
Pulses	34.13	124.99	6.67	0.80	2.16
Other crops	98.26	89.20	19.91	7.32	9.07
Requirements		2,752.56	2,409.00	1,069.70	111.00

This model, which in the interests of simplicity ignores livestock, is as follows:
Maximize $102.85\ x_1 + 114.84\ x_2 + 263.50\ x_3 + 34.13\ x_4 + 98.26\ x_5$ subject to constraints
Land: $773.40\ x_1 + 941.84\ x_2 + 823.13\ x_3 + 124.99\ x_4 + 89.20\ x_5 \leq 2{,}752.56$
Capital: $22.03\ x_1 + 45.74\ x_2 + 63.47\ x_3 + 6.67\ x_4 + 19.91\ x_5 \leq 2{,}409.00$
Labour A: $10.08\ x_1 + 14.10\ x_2 + 17.33\ x_3 + 0.80\ x_4 + 7.32\ x_5 \leq 1{,}069.70$
Labour B: $13.09\ x_1 + 12.07\ x_2 + 18.08\ x_3 + 2.16\ x_4 + 9.07\ x_5 \leq 111.00$
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$.
x_1 = Rice; x_2 = Maize; x_3 = Wheat; x_4 = Pulses; x_5 = Other Crops.
Applying the simplex procedure for obtaining the optimum land of Food Crops through LINGO computer-based software Global optimal solution found.
Objective Value: 1,375.996

Variable	Value	Reduced cost	Rows	Lack or surplus	Dual price
x_1	0.000000	106.5676	1	1375.996	1.000000
x_2	0.000000	102.2864	2	0.000000	0.1050571
x_3	2.565066	0.000000	3	2168.462	0.000000
x_4	0.000000	0.1500135	4	973.0925	0.000000
x_5	7.124985	0.000000	5	0.000000	9.791170

The solution to this model yields the following information: $x_3 = 2.57$ acres of wheat and $x_5 = 7.11$ acres of other food crops. The ultimate aim is to produce realistic agriculture planning model for the regions in order to examine in detail the effect of variations in prices and quantities. <u>The end of citation.</u>

Now we demonstrate Stiefel's method. **In the initial table** (Table 6) **two systems of equations are inscribed.** We offer to interpret such systems as productions functions in a broad sense because of these systems reflect interdependence between different indicators of the definite economic unit.

After two pivot transformations, we get **final table** (Table 7) **which contains two systems properly equivalent to initial systems.** Let us stress that absolutely all numbers in both of tables can be clearly and pithy interpreted as slope coefficients trough their role in the linear equations. We appeal colleagues to provide themselves the proper calculations in order to check correctness of all 11 equations inscribed in the Table 7.

Table 6. The initial table for the optimal land utilizing structure model

0	x_1	x_2	x_3	x_4	x_5	1	
u_1	-773.40	-941.84	-823.13	-124.99	-89.20	2,752.56	y_1
u_2	-22.03	-45.74	-63.47	-6.67	-10.91	2409	y_2
u_3	-10.08	-14.10	-17.33	-0.80	-7.32	1,069.70	y_3
u_4	-13.09	-12.07	-18.08	-2.16	-9.07	111	y_4
Z	102.85	114.84	263.5	34.13	98.26	0	1
	$-v_1$	$-v_2$	$-v_3$	$-v_4$	$-v_5$	w	

Table 7. The final table for the optimal land utilizing structure model

	u_1	x_1	x_2	u_4	x_4	1	
x_3	-0.00155	-0.99898	-1.27555	0.01524	-0.16077	2.573788	v_3
u_2	0.064653	35.39533	21.99719	0.567024	2.635808	2168.098	y_2
u_3	0.004244	3.220009	-0.86584	0.765323	1.383494	973.0686	y_3
x_5	0.003089	0.548141	1.211895	-0.14063	0.082326	7.107597	v_5
z	-0.1048	-106.522	-102.186	-9.80284	-0.14318	1376.586	1
	$-y_1$	$-v_1$	$-v_2$	$-y_4$	$-v_4$	w	

Both problems have a unique solution because z has maximal value and w has minimal value then and only then if all variables in the final expressions of z and w equal zero.

Therefore, $u_1^* = 0$, $u_4^* = 0$, $x_1^* = 0$, $x_4^* = 0$; $v_3^* = 0$, $v_5^* = 0$, $y_2^* = 0$, $y_3^* = 0$.
$z^* = 1{,}376.59 = w^*$; $X^* = (0; 0; 2.5738; 0; 7.1076)^T$, $U^* = (0; 2{,}168.10; 973.07; 0)^T$, $Y^* = (0.1048; 0; 0; 9.8028)^T$, $V^* = (106.522; 102.186; 0; 0.143; 0)^T$.

Application 4. Sensitivity analysis. Serious criticism of Microsoft Solver

Elementary sensitivity analysis means:

A. Constraint $u_i \geq 0$ is substituted with $u_i \geq \lambda$; $i \in \{1, 2, 3, 4\}$, $\lambda \in \mathbf{R}$.
B. Constraint $x_i \geq 0$ is substituted with $x_i \geq \lambda$; $i \in \{1, 2, 3, 4, 5\}$, $\lambda \in \mathbf{R}$.
C. Objective $C \cdot X$ is substituted with $C \cdot X + \mu\, u_i$; $i \in \{1, 2, 3, 4\}$, $\mu \in \mathbf{R}$.
D. Objective $C \cdot X$ is substituted with $C \cdot X + \mu\, x_i$; $i \in \{1, 2, 3, 4, 5\}$, $\mu \in \mathbf{R}$.
We must find $z^*(\lambda)$, $X^*(\lambda)$, $U^*(\lambda)$, $Y^*(\lambda)$, $V^*(\lambda)$ or $z^*(\mu)$, $X^*(\mu)$, $U^*(\mu)$, $Y^*(\mu)$, $V^*(\mu)$.

Example 1. The authors of the paper (Sofi et al., 2015) write: 'Applying the simplex procedure for obtaining the optimum land of Food Crops through LINGO computer-based software reported that value of the variable x_1 equals 0, reduced cost equals 106.57'.

The Microsoft Solver reports that optimal value for rice $x_1^* = 0$ but corresponding reduced cost is −106.52.

It is not our purpose to give in this paper the detail list of comparison between LINGO software and Microsoft Solver. We would like only to show the preferences of the Stiefel's interpretation of the simple table as direct and dual systems of equations.

We are proud to report that our students are able to receive from final table a lot of information about rice production in the concrete land utilizing economy not furnished by Solver.

For example, let us substitute the constraint $x_1 \geq 0$ with $x_1 \geq \lambda$. Than from final table (Table 7) we obtain $u_1^* = 0$, $u_4^* = 0$, $x_1^* = \lambda$, $x_2^* = 0$; $x_4^* = 0$.

$z^* = -106.52\,\lambda + 1{,}376.59$; $X^* = (\lambda;\ 0;\ -0.9990\,\lambda + 2.5738;\ 0;\ 0.5481\,\lambda + 7.1076)^T$,

$U^* = (0;\ 35.3953\,\lambda + 2{,}168.10;\ 3.2200\,\lambda + 973.07;\ 0)^T$,

Remark. We consider here only the direct problem and do not examine the changes in the dual problem.

As far as x_3, u_2, u_3, x_5 remain non-negative this table gives us optimal solution for each value of λ. Solving the system of inequalities

$x_3 = -1.00\,\lambda + 2.57 \geq 0$

$u_2 = 35.40\,\lambda + 2{,}168.10 \geq 0$

$u_3 = 3.22\,\lambda + 973.07 \geq 0$

$x_5 = 0.55\,\lambda + 7.11 \geq 0$,

we get $-12.97 \leq \lambda \leq 2.58$.

So, for each $\lambda \in]-12.97;\ 2.58[$ the unique optimal solution of problem is:

$X^* = (\lambda;\ 0;\ -1.00\,\lambda + 2.57;\ 0;\ 0.55\,\lambda + 7.11)^T$

$U^* = (0;\ 35.3953\,\lambda + 2{,}168.10;\ 3.2200\,\lambda + 973.07;\ 0)^T$

$z^* = -106.52\,\lambda + 1376.59$.

Remark. The border-values $\lambda = -12.97$ and $\lambda = 2.58$ have a special role.

Positive values of x_1 have natural interpretations. However it is possible to interpret pithy also negative values of rice acres.

Let us stress: this result is not included in Microsoft Solver's Sensitivity Report (Table 8). Moreover, there are labels of reduced cost in Solvers report 'Allowable Increase', 'Allowable Decrease', similar than labels of shadow prices, but misleading because of absolutely different sense.

Table 8. Part of the sensitivity report provided by Microsoft Solver

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
Variable Cells						
B	x1	0	-106.52	102.85	106.52	1E+30
Constraints						
Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
H	SUMPRODUCT	2,752.56	0.1048	2,752.56	2,300.95	1,660.92

Example 2. Interpreting the final simplex system as production function what describes post-optimized economics of the investigated unit, we can get answers to different 'what if' questions. Let, for example, $u_1 \geq -10$, $u_4 \geq 20$, $x_2 \geq 5$.
Than we obtain the optimal plan $X^* = (0; 5; 1.31; 0; 8.18)^T$; $z^* = 1264.49$.
Condition $u_1 \geq -10$ means that land is available 2752.56 + 10 = 2762.56 units; condition $u_4 \geq 20$ means that labour B is available 111 − 20 = 91 working hours; condition $x_2 \geq 5$ means that manager voluntary determines to produce 5 units of maize in spite of optimal plan not recommending to produce this product. It is also easy to interpret the values $u_2^* = 2190.73$; $u_3^* = 972.97$.
Important remark. We hope that with help of simple agro metrics example borrowed from article (Sofi et al., 2015) we clearly demonstrate sufficient difference between Dantzig's method and Stiefel's method. The Stiefel's method allows us to interpret the content of Table 6 (Initial table for the optimal land utilizing structure model) and content of Table 7 (Final table for the optimal land utilizing structure model) as two pairs of production functions what characterizes concrete land utilizing economy. Of course, all other tables created during simplex process also can be interpreted as pairs of production functions. Our main assertion is that such interpretations are not possible using traditional Danzig's method.

Our recommendation to Microsoft is to perfect the Solver in order to have available full final simplex table.

Application 5. Vectorial form

Absolutely the same calculations can be interpreted in vectorial form (Table 9 and Table 10). As result we find the useful linear connections between gradient of direct problem's objective C, rows of matrix $A \in \mathbf{R}^{m,n}$, and vectors of standard basis I_k in the space $\mathbf{R}^{n,1}$, and connections between gradient of dual problem's objective B, columns of matrix A, and vectors of standard basis J_k in the space $\mathbf{R}^{m,1}$ as well.

Table 9. The initial table in the vectorial form

	I_1	I_2	I_3	I_4	I_5	O	
$-A_1$	-773.40	-941.84	-823.13	-124.99	-89.20	2,752.56	J_1
$-A_2$	-22.03	-45.74	-63.47	-6.67	-10.91	2409	J_2
$-A_3$	-10.08	-14.10	-17.33	-0.80	-7.32	1,069.70	J_3
$-A_4$	-13.09	-12.07	-18.08	-2.16	-9.07	111	J_4
C	102.85	114.84	263.50	34.13	98.26	0	O
	$-A_1$	$-A_2$	$-A_3$	$-A_4$	$-A_5$	B	

Table 10. The final table in the vectorial form

	$-A_1$	I_1	I_2	$-A_4$	I_4	O	
I_3	-0.0016	-0.9990	-1.2756	0.0152	-0.1608	2.5738	J_3
$-A_2$	0.0647	35.3953	21.9972	0.5670	2.6358	2,168.0980	A_2
$-A_3$	0.0042	3.2200	-0.8658	0.7653	1.3835	973.0686	A_3
I_5	0.0031	0.5481	1.2119	-0.1406	0.0823	7.1076	J_5
C	-0.1048	-106.5220	-102.1860	-9.8028	-0.1432	1,376.5860	O
	$-J_1$	$-A_1$	$-A_2$	$-J_4$	$-A_4$	B	

For example, Table 10 shows that
$A_2 = 0.0647\ A_1 + 0.5670\ A_4 - 35.3953\ I_1 - 2.6358\ I_4$
$A_1 = -35.3953\ A_2 - 3.2200\ A_3 + 0.9990\ J_3 - 0.5481\ J_5$.
These are the agro metric connections between vectors of economic indicators. To interpret these equations we have to keep in mind economical sense of each vector. For example, let us remember that vector A_2. characterizes capital requirements and vector A_1 characterizes rice production in the concrete land utilizing economy. Coefficients 0.0647; 0.5670; −35.3953; − 3.2200 can be interpreted as slope coefficients in the proper linear equations.

CONCLUSIONS

1. The fact that linearity postulate holds in the interconnections between economic indicators in different areas leads to the wide applications of the linear models for quantitative approach to the decision making in management science.
2. Very important reason for linear planning popularity is also the comfortable possibility to solve big scale linear problems with help of effective software.
3. It is easy to observe that the linear programming mainstream applies Dantzig's simplex method, but Stiefel's method is in some kind of oblivion.
4. Of course, Dantzig's method and Stiefel's method operates with absolutely the same information and both methods differ only from interpretations point of view. For all that Hegel's dialectics teaches us that form has influence to the content. Indeed, Stiefel's method offers us the new interpretations and even new concepts, for example, the general optimal plan or simplex systems of equations as production functions.
5. The Stiefel's method allows us to interpret the content of each simplex table as two pairs of production functions what characterizes concrete economic unit. Especially fruitful for economic analysis is the content of final table what characterizes concrete economy close by the received proper optimum. Such interpretations of the simplex tables are not possible using the traditional Dantzig's method.
6. Absolutely no doubt that the Stiefel's method has didactical preferences in teaching process. Especially it concerns non-mathematical specialities' students, for example, the faculties of economics or agronomy.

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